

ON HOMOGENEOUS HYPERSURFACES IN  $\mathbb{C}^3$ 

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**ABSTRACT.** We consider a family  $M_t^n$ , with  $n \geq 2$ ,  $t > 1$ , of real hypersurfaces in a complex affine  $n$ -dimensional quadric arising in connection with the classification of homogeneous compact simply-connected real-analytic hypersurfaces in  $\mathbb{C}^n$  due to Morimoto and Nagano. To finalize their classification, one needs to resolve the problem of the embeddability of  $M_t^n$  in  $\mathbb{C}^n$  for  $n = 3, 7$ . In our earlier article we showed that  $M_t^7$  is not embeddable in  $\mathbb{C}^7$  for every  $t$  and that  $M_t^3$  is embeddable in  $\mathbb{C}^3$  for all  $1 < t < 1 + 10^{-6}$ . In the present paper, we improve on the latter result by showing that the embeddability of  $M_t^3$  in fact takes place for  $1 < t < \sqrt{(2 + \sqrt{2})}/3$ . This is achieved by analyzing the explicit totally real embedding of the sphere  $S^3$  in  $\mathbb{C}^3$  constructed by Ahern and Rudin. For  $t \geq \sqrt{(2 + \sqrt{2})}/3$  the problem of the embeddability of  $M_t^3$  remains open.

## 1. INTRODUCTION

For  $n \geq 2$ , consider the  $n$ -dimensional affine quadric in  $\mathbb{C}^{n+1}$ :

$$(1.1) \quad Q^n := \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \dots + z_{n+1}^2 = 1\}.$$

The group  $\mathrm{SO}(n+1, \mathbb{R})$  acts on  $Q^n$ , with the orbits of the action being the sphere  $S^n = Q^n \cap \mathbb{R}^{n+1}$  as well as the compact pairwise CR-nonequivalent strongly pseudoconvex hypersurfaces

$$(1.2) \quad M_t^n := \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_1|^2 + \dots + |z_{n+1}|^2 = t\} \cap Q^n, \quad t > 1,$$

which are simply-connected for  $n \geq 3$ . The codimension 1 orbits  $M_t^n$  play an important role in the classical paper [MN] whose main objective was the determination of all compact simply-connected real-analytic hypersurfaces in  $\mathbb{C}^n$  homogeneous under an action of a Lie group by CR-transformations. Namely, it was shown in [MN] that every such hypersurface is CR-equivalent to either the sphere  $S^{2n-1}$  or, for  $n = 3, 7$ , to the manifold  $M_t^n$  for some  $t$ . However, the question of the existence of a real-analytic CR-embedding of  $M_t^n$  in  $\mathbb{C}^n$  for  $n = 3, 7$  was not resolved, thus the classification in these two dimensions was not completed.

The family  $M_t^n$  was studied in our earlier paper [I]. In particular, in [I, Corollary 2.1] we observed that a necessary condition for the existence of a real-analytic CR-embedding of  $M_t^n$  in  $\mathbb{C}^n$  is the embeddability of the sphere  $S^n$  in  $\mathbb{C}^n$  as a totally real submanifold. The problem of the existence of a totally real embedding of  $S^n$  in  $\mathbb{C}^n$  was considered by Gromov (see [G1] and [G2, p. 193]), Stout-Zame (see [SZ]), Ahern-Rudin (see [AR]), Forstnerič (see [F1], [F2], [F3]). It has turned out that  $S^n$  admits a smooth totally real embedding in  $\mathbb{C}^n$  only for  $n = 3$ , hence [I, Corollary 2.1] implies, in particular, that  $M_t^7$  cannot be real-analytically CR-embedded in  $\mathbb{C}^7$ . On the other hand, since  $S^3$  is a totally real submanifold of  $Q^3$ , any real-analytic totally real embedding of  $S^3$  in  $\mathbb{C}^3$  (which is known to exist, for instance, by [AR]) extends to a biholomorphic map defined in a neighborhood of  $S^3$  in  $Q^3$ . Owing to the fact that  $M_t^3$  accumulate to  $S^3$  as  $t \rightarrow 1$ , this observation implies

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that  $M_t^3$  admits a real-analytic CR-embedding in  $\mathbb{C}^3$  for all  $t$  sufficiently close to 1. Thus, the classification of homogeneous compact simply-connected real-analytic hypersurfaces in complex dimension 3 is special as it includes manifolds other than the sphere  $S^5$ .

More precisely, in [I, Theorem 3.1] we showed that  $M_t^3$  embeds in  $\mathbb{C}^3$  for all  $1 < t < 1 + 10^{-6}$ . This was proved by analyzing the holomorphic continuation  $F : \mathbb{C}^4 \rightarrow \mathbb{C}^3$  of the explicit polynomial totally real embedding of  $S^3$  in  $\mathbb{C}^3$  constructed in [AR]. Also, in [I, Conjecture 3.1] we stated that the map  $F$  should in fact yield an embedding for all  $1 < t < \sqrt{(2 + \sqrt{2})/3}$ . In the present paper we confirm the conjecture and therefore obtain the following result:

**THEOREM 1.1.** *The hypersurfaces  $M_t^3$  admit a real-analytic CR-embedding in  $\mathbb{C}^3$  for  $1 < t < \sqrt{(2 + \sqrt{2})/3}$ .*

Before proceeding with the proof of the theorem we note that, although the problem of the embeddability of  $M_t^3$  remains open for  $t \geq \sqrt{(2 + \sqrt{2})/3}$ , one might be able to approach it by utilizing other totally real embeddings of  $S^3$  in  $\mathbb{C}^3$  found in [AR] (see Remark 3.3).

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## 2. PROOF OF THEOREM 1.1

As stated in the introduction, our argument is based on analyzing the holomorphic continuation of the explicit totally real embedding of  $S^3$  in  $\mathbb{C}^3$  constructed in [AR]. Let  $(z, w)$  be coordinates in  $\mathbb{C}^2$  and let  $S^3$  be realized in the standard way as the subset of  $\mathbb{C}^2$  given by

$$S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}.$$

The Ahern-Rudin map, which embeds  $S^3$  in  $\mathbb{C}^3$  as a totally real submanifold, is defined on all of  $\mathbb{C}^2$  as follows:

$$(2.1) \quad f : \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad f(z, w) := (z, w, w\bar{z}w^2 + iz\bar{z}^2\bar{w}).$$

Now, consider  $\mathbb{C}^4$  with coordinates  $z_1, z_2, z_3, z_4$  and embed  $\mathbb{C}^2$  in  $\mathbb{C}^4$  as the totally real subspace  $\mathbb{R}^4$ :

$$(z, w) \mapsto (\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w, \operatorname{Im} w).$$

Clearly, the push-forward of the polynomial map  $f$  extends from  $\mathbb{R}^4$  to a holomorphic map  $F : \mathbb{C}^4 \rightarrow \mathbb{C}^3$  by the formula

$$(2.2) \quad F(z_1, z_2, z_3, z_4) := \begin{pmatrix} z_1 + iz_2, z_3 + iz_4, \\ (z_3 + iz_4)(z_1 - iz_2)(z_3 - iz_4)^2 + i(z_1 + iz_2)(z_1 - iz_2)^2(z_3 - iz_4) \end{pmatrix}.$$

It will be convenient for us to argue in the coordinates

$$(2.3) \quad w_1 := z_1 + iz_2, \quad w_2 := z_1 - iz_2, \quad w_3 := z_3 + iz_4, \quad w_4 := z_3 - iz_4.$$

In these coordinates the quadric  $Q^3$  takes the form

$$(2.4) \quad \{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : w_1w_2 + w_3w_4 = 1\}$$

(see (1.1)), the sphere  $S^3 \subset Q^3$  the form

$$\{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : w_2 = \bar{w}_1, w_4 = \bar{w}_3\} \cap Q^3,$$

the hypersurface  $M_t^3 \subset Q^3$  the form

$$(2.5) \quad \{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : |w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 = 2t\} \cap Q^3$$

(see (1.2)), and the map  $F$  the form

$$(2.6) \quad (w_1, w_2, w_3, w_4) \mapsto (w_1, w_3, w_2 w_3 w_4^2 + i w_1 w_2^2 w_4)$$

(see (2.2)).

Clearly,  $F$  yields an embedding of  $M_t^3$  in  $\mathbb{C}^3$  if its restriction  $\tilde{F} := F|_{Q^3}$  is nondegenerate and injective on  $M_t^3$ , therefore it is important to investigate the nondegeneracy and injectivity properties of  $\tilde{F}$ .

Let us start with nondegeneracy. In [1, Proposition 3.1] we gave a necessary and sufficient condition for  $\tilde{F}$  to degenerate somewhere on  $M_t^3$ . As the argument is quite short, we repeat it here—with additional details—for the sake of the completeness of our exposition.

**Proposition 2.1.** *The map  $\tilde{F}$  degenerates at some point of  $M_t^3$  if and only if  $t \geq \sqrt{(2 + \sqrt{2})}/3$ .*

*Proof.* Observe that  $|w_1| + |w_3| > 0$  on  $Q^3$ . For  $w_1 \neq 0$  we choose  $w_1, w_3, w_4$  as local coordinates on  $Q^3$  and write the third component of  $\tilde{F}$  as

$$\varphi := \frac{1 - w_3 w_4}{w_1} (i w_4 + (1 - i) w_3 w_4^2)$$

(see (2.4), (2.6)). Then the Jacobian  $J_{\tilde{F}}$  of  $\tilde{F}$  is equal to

$$(2.7) \quad \frac{\partial \varphi}{\partial w_4} = \frac{(3i - 3)w_3^2 w_4^2 + (2 - 4i)w_3 w_4 + i}{w_1},$$

hence it vanishes if and only if

$$(2.8) \quad w_3 w_4 = \frac{3 \pm \sqrt{2} - i}{6}.$$

At such points using (2.4) we obtain

$$(2.9) \quad |w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 = |w_1|^2 + \frac{2 \mp \sqrt{2}}{6|w_1|^2} + |w_3|^2 + \frac{2 \pm \sqrt{2}}{6|w_3|^2}.$$

Analogously, if  $w_3 \neq 0$ , we choose  $w_1, w_2, w_3$  as local coordinates on  $Q^3$  and write the third component of  $\tilde{F}$  as

$$\psi := \frac{1 - w_1 w_2}{w_3} (w_2 + (i - 1) w_1 w_2^2)$$

(see (2.4), (2.6)). Then

$$J_{\tilde{F}} = -\frac{\partial \psi}{\partial w_2} = -\frac{(3 - 3i)w_1^2 w_2^2 + (2i - 4)w_1 w_2 + 1}{w_3},$$

which vanishes if and only if

$$w_1 w_2 = \frac{3 \pm \sqrt{2} + i}{6}.$$

Hence for all points of degeneracy of  $\tilde{F}$  we have  $w_1 \neq 0$ ,  $w_3 \neq 0$ , and therefore such points are described as the zeroes of  $\partial \varphi / \partial w_4$  or, equivalently, as the zeroes of  $\partial \psi / \partial w_2$ .

We now need the following elementary lemma, which we state without proof:

**Lemma 2.2.** *For fixed  $p > 0$ ,  $q > 0$ , let*

$$g(x, y) := x + \frac{p}{x} + y + \frac{q}{y}, \quad x > 0, y > 0.$$

*Then  $\min_{x>0, y>0} g(x, y) = 2(\sqrt{p} + \sqrt{q})$ .*

Set  $p = (2 + \sqrt{2})/6$ ,  $q = (2 - \sqrt{2})/6$  and observe that  $\sqrt{p} + \sqrt{q} = \sqrt{(2 + \sqrt{2})/3}$ . From formulas (2.5), (2.9) combined with Lemma 2.2 one now easily deduces that if  $J_{\tilde{F}}$  vanishes at a point of  $M_t^3$ , then  $t \geq \sqrt{(2 + \sqrt{2})/3}$ . Conversely, for any  $t \geq \sqrt{(2 + \sqrt{2})/3}$  there exist  $x_0 > 0$ ,  $y_0 > 0$  such that  $g(x_0, y_0) = 2t$ . It then follows from (2.4), (2.5), (2.8) that the point

$$W_0 := \left( \sqrt{x_0}, \frac{3 + \sqrt{2} + i}{6\sqrt{x_0}}, \sqrt{y_0}, \frac{3 - \sqrt{2} - i}{6\sqrt{y_0}} \right)$$

lies in  $M_t^3$  and that  $J_{\tilde{F}}$  vanishes at  $W_0$ . The proof is complete.  $\square$

Next, we study the fibers of  $\tilde{F}$ .

**Proposition 2.3.** *Let two points  $W = (w_1, w_2, w_3, w_4)$  and  $\hat{W} = (\hat{w}_1, \hat{w}_2, \hat{w}_3, \hat{w}_4)$  lie in  $Q^3$  and assume that  $\tilde{F}(W) = \tilde{F}(\hat{W})$ . Then the following holds:*

- (a)  $\hat{w}_1 = w_1$ ,  $\hat{w}_3 = w_3$ ;
- (b) if  $w_1 = 0$  or  $w_3 = 0$ , then  $\hat{W} = W$ ;
- (c) if  $w_1 \neq 0$  and  $w_3 \neq 0$ , then either  $\hat{W} = W$  or

$$(2.10) \quad \hat{w}_4 = \frac{2i - 1 + (1 - i)w_3w_4 + \sqrt{6iw_3^2w_4^2 - (2 + 6i)w_3w_4 + 1}}{(2i - 2)w_3},$$

- (d) neither of the two values in the right-hand side of (2.10) is equal to  $w_4$  if  $W \in M_t^3$  for  $t < \sqrt{(2 + \sqrt{2})/3}$ ;
- (e) the two values in the right-hand side of (2.10) are distinct if  $W \in M_t^3$  for  $t < \sqrt{(2 + \sqrt{2})/3}$ .

Hence, the fiber  $\tilde{F}^{-1}(\tilde{F}(W))$  consists of at most three points, and, if  $W \in M_t^3$  with  $w_1 \neq 0$ ,  $w_3 \neq 0$  for  $t < \sqrt{(2 + \sqrt{2})/3}$ , it consists of exactly three points.

*Proof.* Part (a) is immediate from (2.6). Furthermore, (2.6) yields

$$(2.11) \quad \hat{w}_2w_3\hat{w}_4^2 + iw_1\hat{w}_2^2\hat{w}_4 = w_2w_3w_4^2 + iw_1w_2^2w_4,$$

which together with (2.4) implies part (b).

From now on, we assume that  $w_1 \neq 0$  and  $w_3 \neq 0$ . Then using (2.4) we substitute

$$(2.12) \quad w_2 = \frac{1 - w_3w_4}{w_1}, \quad \hat{w}_2 = \frac{1 - w_3\hat{w}_4}{w_1}$$

into (2.11) and simplifying the resulting expression obtain

$$(2.13) \quad (\hat{w}_4 - w_4) \left[ (i - 1)w_3^2\hat{w}_4^2 + \left( (1 - 2i)w_3 + (i - 1)w_3^2w_4 \right) \hat{w}_4 + \left( i + (1 - 2i)w_3w_4 + (i - 1)w_3^2w_4^2 \right) \right] = 0.$$

We treat identity (2.13) as an equation with respect to  $\hat{w}_4$ . By part (a) and formula (2.12), the solution  $\hat{w}_4 = w_4$  leads to the point  $W$ . Further, the solutions of the quadratic equation

$$(2.14) \quad (i - 1)w_3^2\hat{w}_4^2 + \left( (1 - 2i)w_3 + (i - 1)w_3^2w_4 \right) \hat{w}_4 + \left( i + (1 - 2i)w_3w_4 + (i - 1)w_3^2w_4^2 \right) = 0$$

are given by formula (2.10). This establishes part (c).

Next, for  $\hat{w}_4 = w_4$  equation (2.14) becomes

$$(3i - 3)w_3^2w_4^2 + (2 - 4i)w_3w_4 + i = 0,$$

which by formula (2.7) implies that the Jacobian  $J_{\tilde{F}}$  vanishes at  $W$ . Therefore, part (d) follows from Proposition 2.1.

Finally, suppose that  $6iw_3^2w_4^2 - (2 + 6i)w_3w_4 + 1 = 0$ . Then we have

$$w_3w_4 = \frac{3 \pm 2\sqrt{2} - i}{6}.$$

At such points using (2.12) we obtain

$$(2.15) \quad |w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 = |w_1|^2 + \frac{3 \mp 2\sqrt{2}}{6|w_1|^2} + |w_3|^2 + \frac{3 \pm 2\sqrt{2}}{6|w_3|^2}.$$

From formulas (2.5), (2.15) combined with Lemma 2.2 for  $p = (3 + 2\sqrt{2})/6$ ,  $q = (3 - 2\sqrt{2})/6$  one now deduces that  $t \geq 2/\sqrt{3}$ . Since  $2/\sqrt{3} > \sqrt{(2 + \sqrt{2})/3}$ , part (e) follows. The proof is complete.  $\square$

Propositions 2.1, 2.3 and formula (2.12) imply that in order to establish Theorem 1.1 it suffices to show that for every value  $t < \sqrt{(2 + \sqrt{2})/3}$  and every point  $W = (w_1, (1 - w_3w_4)/w_1, w_3, w_4) \in M_t^3$  with  $w_1 \neq 0$ ,  $w_3 \neq 0$ , the point  $\hat{W} := (w_1, (1 - w_3\hat{w}_4)/w_1, w_3, \hat{w}_4)$  does not lie in  $M_t^3$  for any of the two choices of  $\hat{w}_4$  in (2.10).

Let

$$(2.16) \quad \mathcal{E} := \left\{ z \in \mathbb{C} : |1 - z| + |z| < \sqrt{(2 + \sqrt{2})/3} \right\}.$$

Clearly,  $\mathcal{E}$  is the domain in  $\mathbb{C}$  bounded by the ellipse

$$\left\{ z \in \mathbb{C} : |1 - z| + |z| = \sqrt{(2 + \sqrt{2})/3} \right\},$$

which has foci at 1 and 0. We will need the following lemma:

**Lemma 2.4.** *If a point  $W = (w_1, (1 - w_3w_4)/w_1, w_3, w_4)$  with  $w_1 \neq 0$ ,  $w_3 \neq 0$  lies in  $M_t^3$  for some  $t < \sqrt{(2 + \sqrt{2})/3}$ , then  $w_3w_4 \in \mathcal{E}$ . Moreover, for every  $z \in \mathcal{E}$  there exists  $t < \sqrt{(2 + \sqrt{2})/3}$  and  $W = (w_1, (1 - w_3w_4)/w_1, w_3, w_4) \in M_t^3$  with  $w_1 \neq 0$ ,  $w_3 \neq 0$  such that  $z = w_3w_4$ .*

*Proof.* Fix  $W = (w_1, (1 - w_3w_4)/w_1, w_3, w_4) \in M_t^3$  with  $w_1 \neq 0$ ,  $w_3 \neq 0$  and let  $a := w_3w_4$ . Then (2.5) yields

$$(2.17) \quad |w_1|^2 + \frac{|1 - a|^2}{|w_1|^2} + |w_3|^2 + \frac{|a|^2}{|w_3|^2} = 2t.$$

By Lemma 2.2 with  $p = |1 - a|^2$ ,  $q = |a|^2$ , the left-hand side of (2.17) is bounded from below by  $2(|1 - a| + |a|)$ . Hence, if  $t < \sqrt{(2 + \sqrt{2})/3}$ , one has  $a \in \mathcal{E}$ .

Now, fix  $z \in \mathcal{E}$  and assume first that  $z \neq 0, 1$ . Set  $t = |1 - z| + |z|$  and  $W = (\sqrt{1 - z}, \sqrt{1 - z}, \sqrt{z}, \sqrt{z})$ , where we choose one of the two possible values of the square root arbitrarily. In each of the remaining cases  $z = 0$  and  $z = 1$  we let  $t = 1.045$  and set  $W = (1, 1, 0.3, 0)$  and  $W = (0.3, 0, 1, 1)$ , respectively. It is then clear that  $t < \sqrt{(2 + \sqrt{2})/3}$ , the point  $W$  lies in  $M_t^3$ , the first and third coordinates of  $W$  are nonzero, and the product of the third and fourth coordinates of  $W$  is equal to  $z$ .  $\square$

We will now finalize the proof of the theorem. Fix  $t < \sqrt{(2 + \sqrt{2})/3}$  and a point  $W = (w_1, (1 - w_3w_4)/w_1, w_3, w_4) \in M_t^3$  with  $w_1 \neq 0$ ,  $w_3 \neq 0$ . Consider the point

$\hat{W} := (w_1, (1 - w_3 \hat{w}_4)/w_1, w_3, \hat{w}_4)$  for some choice of  $\hat{w}_4$  in (2.10). Assume that  $\hat{W} \in M_t^3$  and let  $a := w_3 w_4$ ,  $\hat{a} := w_3 \hat{w}_4$ . Then by (2.10) we see

$$(2.18) \quad \hat{a} = \frac{2i - 1 + (1 - i)a + \sqrt{6ia^2 - (2 + 6i)a + 1}}{2i - 2}.$$

Also, recalling that formula (2.10) is derived from equation (2.14), we have

$$(2.19) \quad (i - 1)\hat{a}^2 + (1 - 2i)\hat{a} + (i - 1)\hat{a}a + (i - 1)a^2 + (1 - 2i)a + i = 0.$$

Next, analogously to (2.17), by (2.5) the following holds:

$$(2.20) \quad \begin{aligned} |w_1|^2 + \frac{|1 - a|^2}{|w_1|^2} + |w_3|^2 + \frac{|a|^2}{|w_3|^2} &= 2t, \\ |w_1|^2 + \frac{|1 - \hat{a}|^2}{|w_1|^2} + |w_3|^2 + \frac{|\hat{a}|^2}{|w_3|^2} &= 2t. \end{aligned}$$

Subtracting one equation in (2.20) from the other yields

$$(2.21) \quad \frac{|1 - a|^2 - |1 - \hat{a}|^2}{|w_1|^2} + \frac{|a|^2 - |\hat{a}|^2}{|w_3|^2} = 0.$$

Let us now show that neither of the numerators in (2.21) is zero. Indeed, the vanishing of any one of them implies the vanishing of the other, which can only occur if either  $\hat{a} = a$  or  $\hat{a} = \bar{a}$ . The first possibility means that  $\hat{w}_4 = w_4$  contradicting part (d) of Proposition 2.3. Assuming that  $\hat{a} = \bar{a}$ , from identity (2.19) we deduce

$$(2.22) \quad (2i - 2)\operatorname{Re}(a^2) + (2 - 4i)\operatorname{Re} a + (i - 1)|a|^2 + i = 0.$$

Separating the real and imaginary parts of (2.22) and adding up the resulting identities, we see that  $\operatorname{Re} a = 1/2$ , which leads to  $(\operatorname{Im} a)^2 + 1/4 = 0$  thus implying a contradiction in this case as well.

Hence, by (2.21) we have

$$|w_1|^2 = \frac{|1 - a|^2 - |1 - \hat{a}|^2}{|\hat{a}|^2 - |a|^2} |w_3|^2.$$

Plugging this expression into any of the identities in (2.20) and simplifying the resulting formulas, we obtain the following quadratic equation with respect to  $|w_3|^2$ :

$$(2.23) \quad A|w_3|^4 - 2t|w_3|^2 + B = 0,$$

where

$$(2.24) \quad A := \frac{2(\operatorname{Re} \hat{a} - \operatorname{Re} a)}{|\hat{a}|^2 - |a|^2}, \quad B := \frac{(1 - 2\operatorname{Re} a)|\hat{a}|^2 - (1 - 2\operatorname{Re} \hat{a})|a|^2}{|1 - a|^2 - |1 - \hat{a}|^2}.$$

Taking into account that  $t < \sqrt{(2 + \sqrt{2})}/3$ , for the discriminant  $\Delta$  of the quadratic in the left-hand side of (2.23) we have

$$\Delta = 4(t^2 - AB) < 4\left(\frac{2 + \sqrt{2}}{3} - AB\right).$$

Notice now that, if we let  $t$  vary in the interval  $\left(1, \sqrt{(2 + \sqrt{2})}/3\right)$  and  $W$  in the hypersurface  $M_t^3$ , expressions (2.18) and (2.24) yield an explicit formula for the product  $AB$  viewed as a (two-valued) function of  $a$  where, by Lemma 2.4, the parameter  $a$  varies in the domain  $\mathcal{E}$  defined in (2.16). A straightforward albeit tedious calculation, which we omit because of its length, now implies that everywhere on  $\mathcal{E}$  one has  $AB \geq (2 + \sqrt{2})/3$ . It then follows that  $\Delta < 0$ , which contradicts the existence of a real (in fact, positive) root of equation (2.23).

The proof of Theorem 1.1 is complete.

## 3. A FEW REMARKS

We conclude the paper with several remarks.

*Remark 3.1.* By [I, Proposition 2.1], any real-analytic CR-embedding of  $M_t^n$  in  $\mathbb{C}^n$  extends to a biholomorphic mapping of the domain

$$\{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_1|^2 + \dots + |z_{n+1}|^2 < t\} \cap Q^n$$

onto a domain in  $\mathbb{C}^n$ , where  $n \geq 2$ ,  $t > 1$ . Therefore, Theorem 1.1 implies that for every value  $t < \sqrt{(2 + \sqrt{2})}/3$  and every point  $W = (w_1, (1 - w_3 w_4)/w_1, w_3, w_4)$  in  $M_t^3$  with  $w_1 \neq 0$ ,  $w_3 \neq 0$ , the point  $\hat{W} := (w_1, (1 - w_3 \hat{w}_4)/w_1, w_3, \hat{w}_4)$  lies in  $M_{t'}^3$  where  $t' \geq \sqrt{(2 + \sqrt{2})}/3$  for each choice of  $\hat{w}_4$  in (2.10).

*Remark 3.2.* It is easy to see that the injectivity of  $\tilde{F}$  on  $M_t^3$  fails for all sufficiently large values of  $t$ . For example, if  $t \geq \sqrt{2}$ , let  $u \neq 0$  be a real number satisfying

$$2u^2 + \frac{1}{u^2} = 2t,$$

(cf. Lemma 2.2) and consider the following three distinct points in  $Q^3$ :

$$(3.1) \quad W_u := \left(u, \frac{1}{u}, u, 0\right), W'_u := \left(u, 0, u, \frac{1}{u}\right), W''_u := \left(u, \frac{1+i}{2u}, u, \frac{1-i}{2u}\right).$$

Then  $W_u, W'_u, W''_u \in M_t^3$  and  $\tilde{F}(W_u) = \tilde{F}(W'_u) = \tilde{F}(W''_u) = (u, u, 0)$ . Since by Proposition 2.3 every fiber of  $\tilde{F}$  contains at most three points,  $W_u, W'_u, W''_u$  form the complete fiber of  $\tilde{F}$  over  $(u, u, 0)$ .

*Remark 3.3.* In [AR] the authors in fact introduced not just the map  $f$  (see (2.1)) but a class of maps of the form

$$g : \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad g(z, w) := (z, w, P(z, \bar{z}, w, \bar{w})).$$

Here  $P$  is a harmonic polynomial given by

$$P = \left(\bar{z} \frac{\partial}{\partial w} - \bar{w} \frac{\partial}{\partial z}\right) \left(\sum_{j=1}^m \frac{1}{p_j(q_j + 1)} Q_j\right),$$

where  $Q_j$  is a homogeneous harmonic complex-valued polynomial on  $\mathbb{C}^2$  of total degree  $p_j \geq 1$  in  $z, w$  and total degree  $q_j$  in  $\bar{z}, \bar{w}$ , such that the sum  $Q := Q_1 + \dots + Q_m$  does not vanish on  $S^3$ . It is possible that one can investigate the embeddability of  $M_t^3$  in  $\mathbb{C}^3$  for  $t \geq \sqrt{(2 + \sqrt{2})}/3$  using these more general maps. Note, however, that while it is tempting to take  $Q$  to be a polynomial in  $|z|^2, |w|^2$  (as was done in [AR]), one should avoid doing so as otherwise the resulting polynomial  $P$  would be divisible by  $\bar{z}\bar{w}$ , which implies that the holomorphic extension  $G$  of the push-forward of  $g$  to  $\mathbb{R}^4$  is not injective on  $M_t^3$  with  $t \geq \sqrt{2}$  (cf. Remark 3.2). Indeed, writing  $G$  in the coordinates  $w_j$  defined in (2.3), for the points  $W_u, W'_u$  introduced in (3.1) one has  $G(W_u) = G(W'_u) = (u, u, 0)$ . Thus, one cannot obtain the embeddability of  $M_t^3$  in  $\mathbb{C}^3$  for  $t \geq \sqrt{2}$  by utilizing any of the maps introduced in [AR], with  $Q$  being a function of  $|z|^2, |w|^2$  alone. On the other hand, for  $Q$  of a more general form an analysis of the holomorphic extension  $G$  of the kind we performed above for the map  $F$  becomes computationally quite challenging.

*Remark 3.4.* As explained in [I, Remark 2.2], every hypersurface  $M_t^n$  is nonspherical. Therefore, every manifold  $M_t^3$  embeddable in  $\mathbb{C}^3$  provides an example of a compact strongly pseudoconvex simply-connected hypersurface in  $\mathbb{C}^3$  without umbilic points. Such hypersurfaces have been known before, but the arguments required to obtain nonumbilicity for them are rather involved. For instance, the proof in

[W] of the fact that every generic ellipsoid in  $\mathbb{C}^n$  for  $n \geq 3$  has no umbilic points relies on the Chern-Moser theory. Note for comparison that an example of a compact strongly pseudoconvex hypersurface in  $\mathbb{C}^2$  having no umbilic points has been constructed only very recently (see [ESZ]).

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